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AN ANALYTICAL METHOD FOR SOLVING THE PROBLEM OF UNSHOCKED CONICAL GAS COMPRESSION†

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The two-dimensional unsteady self-similar problem of unlimited unshocked conical compression of a gas is investigated. A solution is constructed in the form of a characteristic series in the domain bounded by a weak discontinuity and the sonic perturbation front. A recursion system of ordinary differential equations is obtained for the coefficients. A boundary-value problem corresponding to the next approximation is investigated in detail, a fundamental system of solutions is found by analytical methods and its asymptotic behaviour is investigated. Essentially independent solutions are determined and different methods are used to seek a solution of the inhomogeneous equation with the required asymptotic behaviour. An algorithm is constructed to compute gas flows induced by the motion of a piston taking the first terms of the series into consideration. The results are compared with those of computations carried out using the method of characteristics. © 1999 Elsevier Science Ltd. All rights reserved.

Processes of unlimited unshocked compression of an ideal polytropic gas, initially at rest inside a prism or a cone-shaped body, have been investigated using the exact two-dimensional and three-dimensional gas-dynamics equations; control laws for the motion of movable pistons producing unlimited compression have been constructed for mutually compatible isentropic exponent and initial geometric parameters of a compressible gas [1–3]. A solution has in fact been obtained for a simplified linear equation describing the flow in the vicinity of the axis of rotation after a weak discontinuity [4].

In this paper we continue the detailed study of the structure of the solution of the Goursat problem in a domain bounded by a weak discontinuity of conical form and a sonic perturbation front. A special change of variables enables us to construct the solution in the form of a characteristic series.

The numerical solution of the Goursat problem using the method of characteristics [3] involves considerable difficulties (the fact that the integration domain is unbounded, the substantial rotation of the characteristics, etc.). It is therefore important to construct a reliable algorithm to compute the gas-dynamic parameters on the basis of the exact solutions.

1. STATEMENT OF THE PROBLEM

Suppose that at time $t = 0$ a uniform polytropic gas with equation of state $p = a^2\rho^\gamma$ (a^2 = const, p is the pressure, ρ is the density and γ is the isentropic exponent and $p = p_0$ = const, $\rho = \rho_0$ = const is at rest inside a body of revolution with generatrix ABO (the axis of revolution is the z axis, r is the radial coordinate, $|\,OB\,| = 1$ and $OB \perp AB$)). We will assume that the initial velocity of sound is $c_0 = 1$. The curve ABO corresponds to the initial position S_t of a movable piston, which begins to advance into the gas at zero initial normal velocity. A weak straight discontinuity GH separates from the piston S_t, propagates at the velocity of sound and separates the domain of rest GHO from the domain of perturbed flow DGHFE. The curve DEF'H corresponds to the piston position at an arbitrary instant of time. The law of motion of S_t must be defined in such a way that, in a process of adiabatic compression with constant entropy, the entire gas at time $t = 1$ is focussed to the point O. The line GE corresponds to the weak discontinuity surface.

In the perturbed domain, the flow is irrotational. The equation for the velocity potential $\Phi(t, z, r)$ has the form

$$\Phi_{tt} + 2\Phi_r\Phi_{rt} + 2\Phi_z\Phi_{zt} + 2\Phi_r\Phi_z\Phi_{rz} + \Phi_r^2\Phi_{rr} + \Phi_z^2\Phi_{zz} - \theta(\Phi_{rr} + \Phi_{zz} + r^{-1}\Phi_r) = 0$$

$$\theta = c^2 = (\gamma - 1)(K - \Phi_t - \Phi_r^2/2 - \Phi_z^2/2)$$

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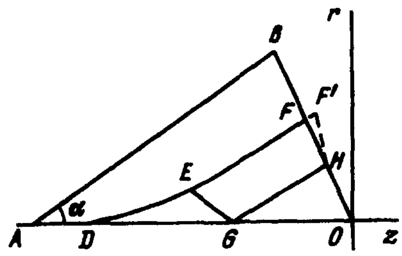

Fig. 1.

where c is the velocity of sound, K = const, $u_r = \Phi_r$, $u_z = \Phi_z$, u_z and u_r are the components of the velocity vector.

The solution to the problem will be constructed in the class of unsteady self-similar conical flows with independent variables

$$\xi = z/\tau, \quad \eta = r/\tau, \quad \tau = t - 1, \quad t \in [0, 1]$$

This class of solutions is described by the equation

$$\begin{aligned}&(\Psi_\xi + \xi)^2 \Psi_{\xi\xi} + 2(\Psi_\xi + \xi)(\Psi_\eta + \eta)\Psi_{\xi\eta} + (\Psi_\eta + \eta)^2 \Psi_{\eta\eta} - \\ &-(\gamma - 1)(\Psi - \xi\Psi_\xi - \eta\Psi_\eta - \Psi_\xi^2/2 - \Psi_\eta^2/2)(\Psi_{\xi\xi} + \Psi_{\eta\eta} + \Psi_\eta/\eta) = 0\end{aligned} \tag{1.1}$$

where $\Phi = Kt - \tau\Psi(\xi, \eta)$.

The flow in the domain DEG has been studied in detail and classes of exact solutions have been constructed [2]. For the case of unlimited cumulation, the function $\Psi(\xi, \eta)$ has the form

$$\Psi = -\frac{2-\gamma}{\gamma+1}\xi^2 - \frac{1}{2}\eta^2 + 3\frac{\gamma-1}{\gamma+1}\xi_0\xi + \frac{(2\gamma-1)}{(3\gamma+1)}\xi_0^2, \quad \xi_0 = \frac{2\sqrt{2-\gamma}}{\sqrt{3}(\gamma-1)} \tag{1.2}$$

In the solution constructed, the angle α is related to the constant γ by

$$\mathrm{tg}\,\alpha = \frac{\sqrt{2-\gamma}}{\sqrt{\gamma+1}}$$

To construct the solution in the domain EGHF, one must solve a Goursat problem with the following data on the characteristics

$$\begin{aligned}&\Psi = \frac{1}{\gamma-1} - \frac{3}{2}\eta^2 \quad &&\text{if} \quad \mu = \xi\sin\alpha + \eta\cos\alpha - 1 = 0 \\ &\Psi = \frac{1}{\gamma-1} &&\text{if} \quad \nu = \xi\sin\alpha - \eta\cos\alpha - 1 = 0\end{aligned} \tag{1.3}$$

It has not been possible to construct an exact solution of this problem using the method of characteristic series [5], because there is a non-analytic singularity in the neighbourhood of the point G. A solution of the Goursat problem has been obtained for a simplified linear equation describing the principal term of the asymptotic expansion of Ψ in the neighbourhood of G in the domain EGH [4]

$$\Psi = \frac{1}{\gamma-1} - \frac{3}{8(\gamma+1)}(3\mu^2 + 2\mu\nu + 3\nu^2)\left(\frac{1}{2} + \frac{1}{\pi}\arcsin\frac{\nu+\mu}{\nu-\mu}\right) - \frac{9}{4(\gamma+1)\pi}\sqrt{-\mu\nu}(\nu+\mu) \tag{1.4}$$

It follows from the form of (1.4) that the function Ψ is not analytic in the neighbourhood of zero. Thus, the nature of the singularity has been determined, and we are interested in further approximations of Ψ.

We change from the variables ξ, η in Eq. (1.1) to the variables ν, μ and then, using our information on the nature of the singularity of Ψ in the neighbourhood of G, we make the substitution

$$y = \arcsin\frac{\nu+\mu}{\nu-\mu}, \quad x=\sqrt{\nu} \tag{1.5}$$

The resulting equation for $\Psi(x, y)$ is not presented here because of its complexity.

With this replacement of the variables, the characteristic $\mu = 0$ transforms into the straight line $y = \pi/2$, and the characteristic $\nu = 0$ transforms into the point $(0, -\pi/2)$ in the OXY plane.

Conditions (1.3) may be written as

$$\Psi=\frac{1}{\gamma-1}-\frac{9}{8(\gamma+1)}\nu^2 \quad \text{if} \quad \mu=0, \quad \Psi=\frac{1}{\gamma-1} \quad \text{if} \quad \nu=0 \tag{1.6}$$

2. CONSTRUCTION OF THE SOLUTION AS A SERIES

We will seek a solution of the Goursat problem as a characteristic series

$$\Psi(x, y)=\sum_{k=0}^{\infty} a_k(y)x^k \tag{2.1}$$

It follows from (1.6) that the coefficients of series (2.1) must satisfy the following conditions

$$a_k\left(\frac{\pi}{2}\right)=0 \quad \text{for any} \quad k\neq 4, \quad a_4\left(\frac{\pi}{2}\right)=-\frac{9}{8(\gamma+1)}$$
$$\lim_{x\to 0} a_k(y)x^k=0 \quad \text{for any} \quad k \tag{2.2}$$

Obviously, $a_0(y) = 1/(\gamma - 1)$. After substituting series (2.1) into the equation for the potential and equating the coefficients of like powers of x to zero, we obtain a system of ordinary differential equations for $a_k(y)$. For $k = 1, 2, 3$ and all further odd k, this gives homogeneous equations which, by (1.6), have only trivial solutions. For $k = 2n$, $n = 2, 3, \ldots$, the equation is

$$\ddot{a}_k+\frac{k\dot{a}_k}{\cos y}-\frac{ka_k}{2(1+\sin y)}=F_k(y) \tag{2.3}$$

The function $F_k(y)$ on the right of Eq. (2.3) depends on $a_s(y)$, $s < k$; $F_4(y) = 0$.

Obviously, Eq. (2.3) has two singular points: $y = -\pi/2$ and $y = \pi/2$. Hence the construction of a solution with the desired asymptotic behaviour is a non-trivial problem. Substituting

$$a_{2n}=u[(1-\sin y)/(1+\sin y)]^{n/2}$$

into (2.3), we obtain an equation for u that does not contain a term with the first derivative

$$\ddot{u}-\frac{n(n+1)}{\cos^2 y}u=0 \tag{2.4}$$

One solution of this equation may be determined by a formula in [6]

$$u_1=\lim_{a\to 0}\cos^n y\left(\frac{1}{\cos y}\frac{\partial}{\partial y}\right)^n e^{ay}$$

The other independent solution may be found by substituting $u_2 = u_1 z$ in (2.4).

For the case $n = 2$, the solution of Eq. (2.3) for $k = 4$ may be written as

$$a_4=\frac{(C_1+C_2y)(2-\cos 2y)+\frac{3}{2}C_2\sin 2y}{(1+\sin y)^2}$$

For the first of conditions (2.2) to be satisfied, the constants C_1 and C_2 must satisfy the relation

$$C_1 + C_2\pi/2 = -\tfrac{3}{2}(\gamma+1)^{-1}$$

Asymptotic analysis of $a_4(y)$ as $y \to -\pi/2$ has shown that if

$$C_1 - C_2\pi/2 = 0$$

then $a_4(y) \to 0$ as $y \to -\pi/2$, and consequently, the second of conditions (2.2) is satisfied. Finally,

$$a_4(y) = -\frac{3}{2\pi(\gamma+1)}\frac{(2-\cos 2y)(\pi/2+y)+\tfrac{3}{2}\sin 2y}{(1+\sin y)^2}$$

Expressing the partial sum of series (2.1) up to and including $a_4(y)$ in terms of the variables ν and μ, we obtain solution (1.4) of the linearized equation, and it is therefore of particular interest to work out the next non-trivial approximation to the solution of the problem.

3. INVESTIGATION OF THE SOLUTION OF THE EQUATION FOR THE SECOND TERM OF THE SERIES

For $n = 3$, Eq. (2.3) has the form

$$\ddot{a}_6 + \frac{6\dot{a}_6}{\cos y} - \frac{3a_6}{(1+\sin y)} = -\frac{36}{\pi^2(\gamma+1)^2(1+\sin y)^3}A(y)$$
$$A(y) = (y^2\sin y + 2y\cos y - \pi^2/4\sin y + \cos^2 y\sin y) \qquad (3.1)$$

Using the algorithm described above, we write down a fundamental system of solutions of the homogeneous equation corresponding to (3.1). We have

$$u_1 = -\frac{\sin y(6+4\sin^2 y)}{\cos^3 y}, \quad u_2 = -\frac{yu_1}{2} + \frac{33\sin^2 y + 12}{9\cos^2 y}$$

Since the function $z(y)$ by which u_1 was multiplied has no singularities at $y = -\pi/2$, it follows that u_1 and u_2 behave in the same way at the ends of the interval $[-\pi/2, \pi/2]$ The general solution of the homogeneous equation has the form $u = C_1u_1 + C_2u_2$, and the constants C_1 and C_2 may be chosen in such a way that the functions in the fundamental system of solutions are essentially independent. In particular, when $C_2 = 4C_1/\pi$, the function u has a singularity in the neighbourhood of $y = -\pi/2$, namely, $u \sim 1/(y+\pi/2)^3$, while in the neighbourhood of the point $y = \pi/2$ we have $u \sim (y+\pi/2)^4$. If we take C_2 equal to $-4C_1/\pi$, then in the neighbourhood of $y = \pi/2$ we have 427m, while in the neighbourhood of $y = \pi/2$ there is a singularity $u \sim (y+\pi/2)^4$. We thus have a fundamental system of solutions of Eq. (2.4) for $n = 3$ and we finally obtain a fundamental system of solutions of the homogeneous equation corresponding to (3.1) in the form

$$\varphi_1 = \zeta^-(y)(1+\sin y)^{-3}, \quad \varphi_2 = \zeta^+(y)(1+\sin y)^{-3} \qquad (3.2)$$
$$\zeta^\pm(y) = \sin y(3+2\sin^2 y)(y \pm \pi/2) + \cos y(11\sin^2 y+4)/3$$

Using the method of variation of constants, we write down the general solution of the inhomogeneous equation (3.1)

$$a_6 = C_1(y)\varphi_1 + C_2(y)\varphi_2, \quad C_{1,2}(y) = \frac{9}{(\gamma+1)^2\pi^3}\int\frac{A(y)\zeta^\pm(y)}{\cos^6 y}dy \qquad (3.3)$$

Since φ_1, φ_2 and the integrands in (3.3) have a singularity at the ends of the interval $[-\pi/2, \pi/2]$, it is important to ascertain whether Eq. (3.1) has a particular solution satisfying conditions (2.2)

$$a_6(\pi/2) = 0, \quad \lim_{x\to 0} a_6(y)x^6 = 0$$

To that end, we made a detailed investigation of the asymptotic behaviour of φ_1, φ_2 and of $a_6(y)$ itself in the neighbourhood of the singular points. It turned out that a solution of Eq. (3.1) exists with the following asymptotic expansions

$$a_6(y)=\frac{36}{35\pi^2(\gamma+1)^2}\left(-\frac{28\pi}{3}\frac{1}{y+\pi/2}+\frac{140}{3}-\frac{141\pi}{35}\left(y+\frac{\pi}{2}\right)\right)+O\left(\left(y+\frac{\pi}{2}\right)^2\right)$$

as $y\to-\pi/2$

$$a_6(y)=-\frac{3}{\pi^2(\gamma+1)^2}\left(\frac{\pi}{20}\left(y-\frac{\pi}{2}\right)^5+\frac{1}{4}\left(y-\frac{\pi}{2}\right)^6+\frac{59\pi}{2800}\left(y-\frac{\pi}{2}\right)^7\right)+O\left(\left(y-\frac{\pi}{2}\right)^8\right)$$

as $y\to\pi/2$

Considering only the principal term of the asymptotic expansion of $a_6(y)$, we investigate the behaviour of the function

$$f(x,y)=a_6(y)x^6=x^6/(y+\pi/2)$$

as $x\to 0$, $y\to-\pi/2$. It is obvious that if

$$y+\pi/2\sim x^\alpha,\quad \alpha\geqslant 6$$

near that point, then $f(x,y)$ does not tend to zero. If we consider $f(x,y)$ in the neighbourhood of the point $x=0$, $y=-\pi/2$ along a curve in the hatched domain of the xy plane (Fig. 2, right), condition (2.2) will be satisfied. Figure 2, left, shows the same domain in the μ, ν plane.

Using the algorithm described here, the process of computing $a_k(y)$ may be continued, obtaining new, more accurate approximations for $\Psi(x,y)$.

4. RESULTS OF NUMERICAL COMPUTATIONS

In order to construct the law of motion of the piston $r=r(t)$, $z=z(t)$, we have to solve a system of ordinary differential equations

$$dr/dt=\Phi_r(r,z,t),\quad dz/dt=\Phi_z(r,z,t)\tag{4.1}$$

subject to the following conditions

$$r=1/\cos\alpha+z\operatorname{tg}\alpha\quad\text{at}\quad t=0$$

By (1.2), the velocity field in the domain DEG is linear with respect to ξ, η and the law of motion of the piston in DEG is described by the following system of equations

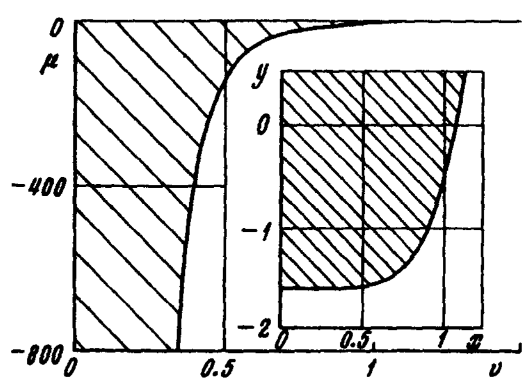

Fig. 2.

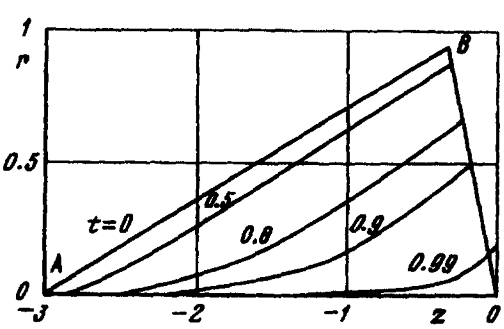

Fig. 3.

Table 1

t	Up to a_4		Up to a_6		Numerical method	
	$-z$	r	$-z$	r	$-z$	r
	2.99903	0.00000	2.99903	0.00000	2.99903	0.00000
	2.50289	0.17535	2.50289	0.17535	2.50288	0.17534
0.05	1.56256	0.50797	1.56254	0.50792	1.56254	0.50790
	0.77111	0.78783	0.77109	0.78776	0.77106	0.78767
	0.33327	0.94265	0.33324	0.94256	0.33318	0.94235
	2.99601	0.00000	2.99601	0.00000	2.99602	0.00000
	2.50226	0.17375	2.50230	0.17383	2.50225	0.17371
0.1	1.56219	0.50697	1.56215	0.50685	1.56211	0.50670
	0.77082	0.78702	0.77073	0.78675	0.77062	0.78639
	0.33300	0.94190	0.33289	0.94156	0.33269	0.94083

$$\frac{dr}{dt}=\frac{r}{t-1},\quad \frac{dz}{dt}=\frac{2(2-\gamma)}{\gamma+1}\frac{z}{t-1}-\frac{2\sqrt{3(2-\gamma)}}{\gamma+1} \tag{4.2}$$

Taking into account the first approximation (1.4) to the function Ψ, we can write down a system of equations describing the law of motion of the piston in the domain EGHF′

$$\begin{gathered}\frac{dr}{dt}=\left(rL+\frac{MN}{\cos^2\alpha}\right)\cos\alpha,\quad \frac{dz}{dt}=-\frac{6}{\gamma+1}(LN+M)\sin\alpha\\ L=\frac{1}{t-1}\left(\frac{1}{2}+\frac{1}{\pi}\arcsin\frac{N}{r\cos a}\right),\quad M=\frac{1}{\pi(t-1)}\sqrt{r^2\cos^2\alpha-N^2}\\ N=t-1-z\sin\alpha\end{gathered} \tag{4.3}$$

At the initial time, the segment AB is partitioned by mesh points converging to the point A. In the domain DEG one can integrate the equations of system (4.2) and write down an exact law of motion of the piston. In the domain EGHF′, system (4.3) cannot be integrated exactly and we therefore use a third-order Runge–Kutta method to obtain a solution.

The velocity of sound $c(t)$, the pressure $p(t)$ and the energy $E(t)$ necessary for unlimited compression are computed at the piston.

Figure 3 shows the form of the moving piston at various instants of time (ABO is the initial position of the piston).

A comparison with the results of computations carried out at the Institute of Mathematics and Mechanics, Ural Department of the Russian Academy of Sciences, by T. N. Bronina, using the method of characteristics, showed that, for large values of t, the laws of motion of the piston are identical to two or three significant figures. The following table lists the coordinates of the points of the piston at t = 0.05 and t = 0., obtained using partial sums of a characteristic series containing four terms and six terms. The coordinates of the points of the piston obtained by the method of characteristics are also listed.

As t increases, the quality of the computations deteriorates. The convergence of the computed expansions was not investigated. Perhaps these series converge only locally, so that one cannot expect good results for t close to unity.

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